

Critical condition of the water-retention model

Seung Ki Baek¹ and Beom Jun Kim^{2,*}

¹*Integrated Science Laboratory, Umeå University, 901 87 Umeå, Sweden[†]*

²*BK21 Physics Research Division and Department of Physics,
Sungkyunkwan University, Suwon 440-746, Korea*

We study how much water can be retained without leaking through boundaries when each unit square of a two-dimensional lattice is randomly assigned a block of unit bottom area but with different heights from zero to $n - 1$. As more blocks are put into the system, there exists a phase transition beyond which the system retains a macroscopic volume of water. We locate the critical points and verify that the criticality belongs to the two-dimensional percolation universality class. If the height distribution can be approximated as continuous for large n , the system is always close to a critical point and the fraction of the area below the resulting water level is given by the percolation threshold. This provides a universal upper bound of areas that can be covered by water in a random landscape.

PACS numbers: 64.60.ah, 47.56.+r, 92.40.Qk

Percolation is a simple yet powerful model used to understand geometric critical phenomena [1, 2]. Studies of gelation [3] and fluid in a porous media [4] go back more than half a century ago. Since then, percolation has been studied in a variety of contexts such as epidemiology [5], ferromagnetism [6], and even microprocessor manufacturing [7]. Recently, it has been pointed out that percolation also may be studied in a geographic context, that is, in determining how much water can be retained without leaking through boundaries on a given rugged landscape [8]. If the landscape has only two levels, the connection to percolation is straightforward. Suppose that we have a square lattice of a certain size. We may define this initial state as level zero. It is more intuitive to start with level one, however, by filling every square site with a cubic block of the unit height. Then, we randomly remove a certain fraction of the cubic blocks, say p_0 , so that we have small ponds here and there where water is retained. As p_0 grows, the ponds merge into large clusters, and when one of these clusters touches the boundary, a large amount of water drains out of the system all at once. Obviously, this crisis happens when p_0 reaches the site-percolation threshold of the square lattice, which has been estimated as $p_c^{\text{site}} \approx 0.592\,746\,02(4)$ [9]; in terms of p_1 , the fraction of blocks with height 1, this corresponds to $p_1 = 1 - p_c^{\text{site}} = 0.407\,253\,98(4)$, which we simply denote as p_c throughout this work. Beyond this direct connection, the water-retention model allows the number of levels to be larger than 2, so we may ask ourselves how these different heights can affect the critical behavior of the system. The distribution of different heights extends the parameter space of the original percolation problem to higher dimensions and thus literally adds a new dimension to percolation, even though

these new parameters turn out to be irrelevant in the renormalization-group sense (see below). It is also notable that the analogy of a random landscape can be found in surface roughening [10] as well as in the concept called a rugged energy landscape used in spin glasses and protein dynamics [11, 12].

In the present work we consider this water-retention problem for a general n -level case where the system may have blocks with heights from 0 to $n - 1$. The fraction p_i of blocks of the height i should satisfy $\sum_{i=0}^{n-1} p_i = 1$ ($p_i \geq 0$), which makes us deal with an $(n-1)$ -dimensional parameter space. A site occupied by a zero-height block is an empty site. We locate critical points in this parameter space and give numerical evidence that the criticality always belongs to the two-dimensional (2D) percolation universality class described by critical exponents $\beta = 5/36$ and $\nu = 4/3$ [13]. The critical condition implies that one can find a universal feature in any random landscape with very large n , which is largely independent of the distribution of blocks $\{p_i\}$.

Our simulation code is based on the burning algorithm. Let us begin with the two-level case for ease of explanation. We have an $(L+2) \times (L+2)$ square lattice, where we assign the boundary state to the outermost squares while the level-zero state is assigned to the other interior sites. At each of these $L \times L$ interior sites, we replace its state with a level-one state with probability p_1 . This step determines the configuration of blocks that we are going to examine in the following way. We pick up a level-zero site from which we run the burning algorithm. The fire can neither penetrate level-one sites nor propagate back. If this fire touches any boundary-state site, the water will drain out. If the burning stops before touching the boundary by being surrounded by level-one blocks, this burned area is added to the amount of water retained by this block configuration. Then we move to another level-zero site that is not burned yet, which we call available, and repeat the same procedure until there are no more such available sites left. One can choose either the depth-first search and the breadth-first search

*Corresponding author, E-mail: beomjun@skku.edu

[†]Present address: School of Physics, Korea Institute for Advanced Study, Seoul 130-722, Korea

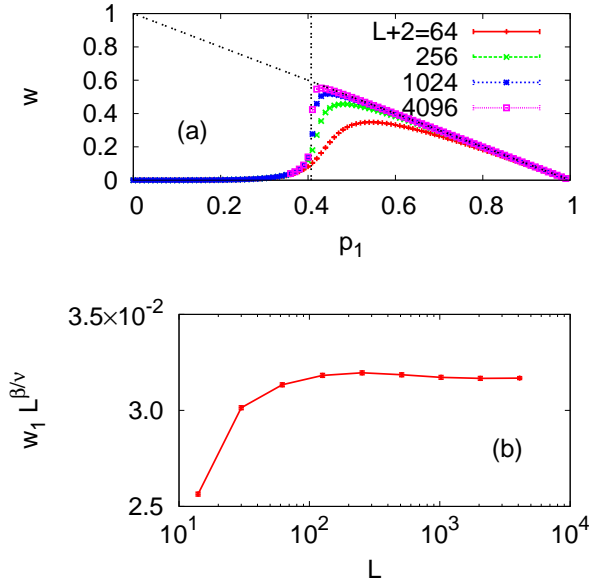


FIG. 1: (Color online) Results of the two-level case. (a) Water volume per area w as a function of p_1 , the fraction of blocks with height 1. The vertical dotted line means $p_c \approx 0.407\ 253\ 98$ and the other line from the top left to the bottom right is for comparing w with $1 - p_1$. (b) $w_1 L^{\beta/\nu}$ at p_c as a function of L , based on Eq. (1), where the number of sample averages is $O(10^5)$ for each data point. Error bars are shown, but they are comparable to the symbol size.

for the burning algorithm. Although the former exhausts memory faster, it is simpler to implement and turns out to have little difficulty in simulating $L \sim 6 \times 10^3$ using Intel Core2 Duo CPU E8400 with 3.00 GHz and 2.0 GB memory. The breadth-first search is reserved for very-large- n cases (see below). Solving a system with blocks of height 0, 1, and 2 requires the system to be burned twice: one occurs at level 1 and the other at level 2. The first burning fills level-zero sites with water up to height 1 precisely as in the previous two-level case. After the first burning, consequently, we are left with sites with levels one and two. The level-two sites must be occupied by blocks of height 2, while the level-one sites can be either occupied by blocks of height 1 or filled with water of that height. When we start the second burning, all these level-one sites should appear available so that we can fill these sites with water of that height. In other words, only blocks with height 2 are of our interest at this second burning. It is straightforward to extend the same procedure to higher levels by raising the water level one by one. Note that we count only such water that does not touch the boundary as retained by the given block configuration every time we run the burning algorithm.

We start from the two-level case of the water-retention problem [8]. As discussed above, it is easier to understand this problem if we start from $p_1 = 1$ and decrease it by adding level-zero sites. One may expect that the amount of water will be roughly proportional to p_0 , espe-

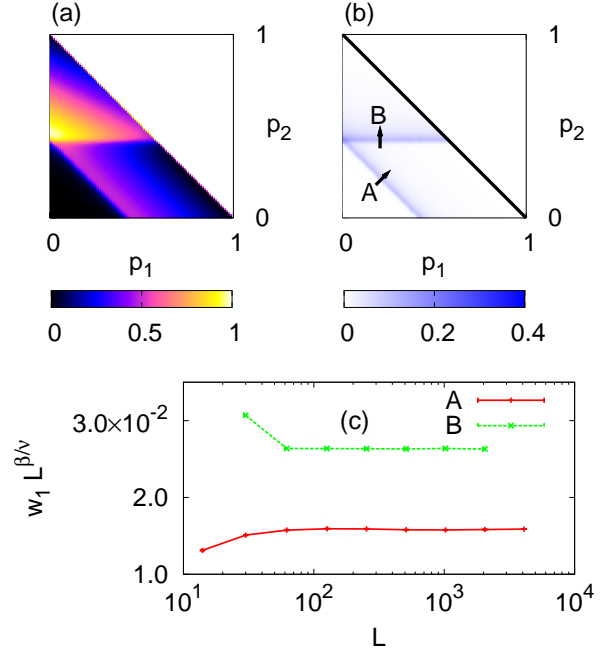


FIG. 2: (Color online) Results of the three-level case. (a) w as a function of p_1 and p_2 at $L + 2 = 512$. (b) Difference in w between $L + 2 = 256$ and 512 . Cross section A, depicted as a slanting arrow, means $p_1 = p_2$, and cross section B, depicted as a vertical arrow, means $p_1 = p_c/2$. (c) $w_1 L^{\beta/\nu}$ measured at $p_1 = p_2 = p_c/2$, the critical point along A, and also at $p_1 = p_2/2 = p_c/2$, the critical point along B. The 2D percolation result $\beta/\nu = 5/48$ is used in common.

cially for $p_0 \ll p_c^{\text{site}}$, as shown in Fig. 1(a). As the system size grows, the linear proportionality becomes more and more accurate because the majority of ponds are centrally located away from the boundary. In Fig. 1(a) the vertical axis means water volume divided by the total interior area $L \times L$ and we denote this observable as w . When p_0 reaches p_c^{site} or, equivalently, when p_1 reaches p_c , percolation occurs and a large amount of water drains out through the boundary. In terms of percolation theory, the observable w corresponds to the sum of sizes of all nonpercolating clusters of level-zero sites in bulk, which will converge to p_c^{site} in the thermodynamic limit [1]. According to the theory of finite-size scaling [1], the largest amount of water in a single pond, denoted by w_1 , is expected to be described by the following form:

$$w_1 = L^{-\beta/\nu} f[(p_1 - p_c)L^{1/\nu}], \quad (1)$$

with $\beta = 5/36$ and $\nu = 4/3$ near criticality [13]. This is readily confirmed by Fig. 1(b), where we plot $w_1 L^{\beta/\nu} \rightarrow \text{const}$ at $p_1 = p_c$.

In the three-level case the result can be depicted in a triangular area restricted by $p_1 + p_2 \leq 1$, as shown in Fig. 2(a), since the fraction of level-zero sites is obviously $p_0 = 1 - p_1 - p_2$ and $p_0 \geq 0$. At each of the three

boundaries of this triangle, we are back to the two-level case. First, if $p_2 = 0$, we move along the low horizontal boundary and the result is exactly Fig. 1(a). Second, if $p_2 = 1 - p_1$, we move along the hypotenuse of the triangle and it is again the two-level problem with level-one and level-two sites because $p_0 = 0$. Finally, if $p_1 = 0$, we move along the left vertical boundary. It is the same pattern as in the two-level case but with double amounts of water since the block height is doubled. The question is then what happens inside these boundaries; Fig. 2(a) gives the answer.

Let us locate critical points inside the triangle. For this purpose we compare two different system sizes in Fig. 2(b). The motivation is as follows. When there are only small ponds around us, one hardly needs to care about how large the system is because everything is local. If the system is close to the critical point so that a large amount of water is about to drain out, however, the correlation length becomes so large that the behavior is essentially constrained by the finite size of the system. In other words, it is around the critical point that our observable w becomes the most sensitive to the system size, as can be seen in Fig. 1(a) for the two-level case. Figure 2(b) indicates that there are two lines of critical points, which can be written as

$$p_1 + p_2 = p_c \quad (2)$$

$$p_2 = p_c. \quad (3)$$

We can argue that these are exact expressions for the following reason: Eq. (2) implies that blocks with height 2 can be substituted for those with height 1, while Eq. (3) implies that the blocks with height 2 have their own critical point without being affected by the presence of level-one blocks.

Another important question is whether the critical property remains the same along these critical lines. The above argument on exactness of Eqs. (2) and (3) implies that this will be the case. We examine two cross sections represented as A and B in Fig. 2(b) and show the results in Fig. 2(c). In both these cases w_1 follows the scaling form of Eq. (1) with $\beta/\nu = 5/48$. We have also observed that the critical region of w scales as $L^{-1/\nu}$ with $\nu = 4/3$ (not shown here). These results strongly suggest that the criticality always belongs to the 2D percolation universality class.

A four-level system requires consideration of the volume inside a tetrahedron, written as $p_1 + p_2 + p_3 \leq 1$. Since it is not easy to visualize such a three-dimensional object, we plot differences in w between two system sizes as before, but now fixing p_3 at 10% in Fig. 3. Comparing this with Fig. 2(b), we immediately see that the new critical surfaces can be written as

$$\begin{aligned} p_1 + p_2 + p_3 &= p_c, \\ p_2 + p_3 &= p_c, \\ p_3 &= p_c. \end{aligned} \quad (4)$$

Note that we had $p_1 = p_c$ as the critical condition for the two-level case and Eqs. (2) and (3) for the three-level

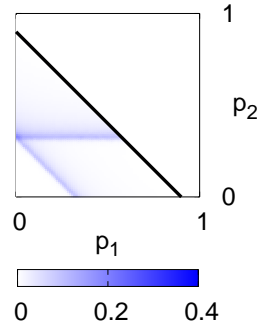


FIG. 3: (Color online) Results of the four-level case. The calculation is the same as Fig. 2(b) except that $p_3 = 0.1$.

case. Comparing Eq. (4) with these previous cases, we can now generalize the critical condition for the n -level case as

$$\begin{aligned} P(1) &= p_c, \\ P(2) &= p_c, \\ &\vdots \\ P(n-1) &= p_c, \end{aligned} \quad (5)$$

where $P(i) \equiv \sum_{j=i}^{n-1} p_j$. Note that the system is critical when *any* of the conditions is satisfied. Let us assume that $P(i)$ can be approximated by a continuous function of i as n gets large. It implies that p_i can be made arbitrarily small by dividing levels ($n \rightarrow \infty$), which is true for most natural landscapes where contour lines do not occupy a finite area fraction. Then one can argue from the intermediate-value theorem that there exists a certain $i = i^*$ that satisfies one of the critical conditions above. In other words, the system is almost always close to a certain critical surface so that the correlation length is found to be very large at a certain water level i^* . Consider a system where a large amount of water is poured onto the model and the excess water is allowed to drain off. For $i \geq i^*$, water will flow out of the system, so the water level should be kept slightly below i^* in a stable situation. Still, $P(i^*)$ is close to p_c and the typical length scale will be very large, which shapes observed ponds and lakes into fractal objects like 2D percolating clusters. In this sense, one may connect this finding to the idea of self-organized criticality [14], although our finding originates from gradual variations in $P(i)$, not from any dynamic process (see also Ref. [15] on self-organized behavior in invasion percolation).

Another implication of Eq. (5) is that the fraction of the area above the observed water level i^* should be always the same for infinitely many $(p_1, p_2, p_3, \dots, p_{n-1})$. The fraction of the area below the water level is given by $p_c^{\text{site}} = 0.592\,746\,02(4)$ in our case, but the numeric value depends on the square geometry we have chosen, so actual field observations may well give a different value.

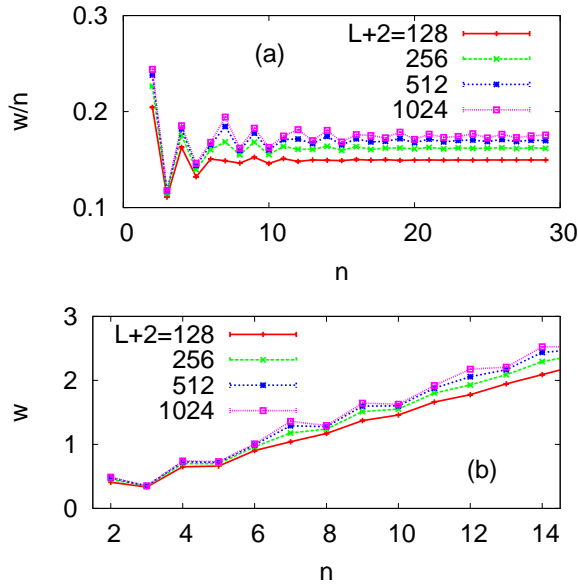


FIG. 4: (Color online) Retained water as a function of n , the number of levels, where $p_i = 1/n$ for every i . (a) As $L \rightarrow \infty$, w/n , with $n \gg 1$, approaches a constant. (b) For finite n , w is not monotonic: $n = 3$ contains a smaller amount than $n = 2$, for example, and such nonmonotonic behavior becomes more visible with larger L .

The important point is that there can be a universal upper limit of how much area can be covered by water at any length scale. Since we have considered randomly distributed blocks, if a landscape contains more water than this upper limit, it indicates the existence of non-random processes with correlation lengths comparable to the scale of our observation.

As a simple example, let us assume that p_i 's are equally distributed, i.e., $p_i = 1/n$ for every i [8]. When n is very large, we have $P(i) \approx 1 - i/n$ and thus $P(i^*) = 1 - i^*/n = p_c$. Water retained in this system roughly amounts to $w = \sum_{i=0}^{i^*} (i^* - i)p_i \approx (1 - p_c)^2 n/2 \approx 0.175 \, 673 \, 9n$. In

fact, this is an underestimate since there can be ponds with water higher than i^* [16]. The point is that w is asymptotically proportional to n [Fig. 4(a)]. However, w might not be a monotonic function of n when n is finite [Fig. 4(b)]. The reason can be found by recalling Eq. (5): The question is how close one can get to a critical condition at each given n while keeping $P(i)$ larger than p_c . For example, one can get $P(1) = 1/2$ for $n = 2$, which differs from p_c by 0.09. For $n = 3$, the minimal difference gets larger because $P(2) - p_c \approx 2/3 - 0.4 \approx 0.26$. It explains the increasing value of $w(n = 2) - w(n = 3)$ with increasing L and the same explanation holds true for larger n 's as well in Fig. 4(b).

In summary, we have considered the general n -level water-retention problem. Specifically, we have found the critical conditions in the $(n - 1)$ -dimensional parameter space and provided numerical evidence that the critical property always belongs to the 2D percolation universality class. The critical condition implies that there are universal features in the large- n limit: First, the system is very likely to be close to a critical point and second, the area below water is not significantly dependent on the distribution of block heights. This tells us how the percolation threshold can be related to the upper limit of water retained by a random landscape and used as a quantitative measure for detecting a geophysical process on a certain length scale when its existence is called into question.[22]

Acknowledgments

We are deeply indebted to Craig L. Knecht for introducing us to this problem, and communication with Robert M. Ziff is gratefully acknowledged. S.K.B. acknowledges the support from the Swedish Research Council with Grant No. 621-2008-4449. B.J.K. was supported by Faculty Research Fund, Sungkyunkwan University. This research was conducted using the resources of High Performance Computing Center North.

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- [22] It is interesting to recall that 70.9% of the Earth is covered by water [17]. This fraction is higher than site-percolation thresholds of the triangular [18], square [9], and honeycomb lattices [19] and also higher than crit-

ical area fractions of 2D continuum percolation models [20, 21]. If we assume that every geophysical process has a short length scale compared to the size of the Earth, so that the global landscape on the Earth can be roughly regarded as random, we may retrospectively understand that too much of the Earth is covered by water to be believed to be a flat disk with steep cliffs at the boundary as once believed.